similarly $x=-2$, $x=4$ are vertical tangents. The equation $\lambda(2\lambda-\mu)+\mu(\lambda-2\mu)=0$ gives $\lambda:\mu = 1:1$ and $1:-1$, the directions of the principal axes ; the equation $\mu(2\lambda-\mu)-\lambda(\lambda-2\mu)=0$ gives $\lambda:\mu = 2+\sqrt{3}:1$ and $2-\sqrt{3}:1$, the directions of the asymptotes.

$$x^2+2xy+y^2-2x-6y+1=0. \tag{3}$$

$\partial f/\partial x = 2(x+y-1)$ and $\partial f/\partial y = 2(x+y-3)$ have the same direction $1:-1$. $-\partial f/\partial x - \partial f/\partial y = 0$ gives $x+y=2$, the axis of the curve. Elimination of x between $f=0$ and $\partial f/\partial x = 0$ gives $y=0$, the horizontal tangent. Similarly $x=2$ is the vertical tangent; these tangents cross on the directrix. The focus may now be located.

$$x^3+2y^3-3xy^2-6x-6y=0. \tag{4}$$

$\partial f/\partial x = 3(x^2-y^2-2)$ and $\partial f/\partial y = 6(y^2-xy-1)$. Lines in the two directions $1:1$ and $2:-1$ cut the cubic in two points only. $\partial f/\partial x + \partial f/\partial y = 0$ gives $(x-y+2)(x-y-2)=0$, both asymptotes of the cubic; $2(\partial f/\partial x)-\partial f/\partial y=0$ gives $x^2+xy-2y^2=1$, a hyperbola whose center is at the origin and whose asymptotes are $x-y=0$ and $x+2y=0$, the latter being also an asymptote of the cubic. This hyperbola bisects all chords having the direction $2:-1$.

A METHOD OF SOLVING NUMERICAL EQUATIONS.

By S. A. COREY, Hiteman, Iowa.

The following development of the roots of an equation by Maclaurin's formula applies to both algebraic and transcendental equations, and gives all the roots approximately whenever the conditions involved in the development can be fully complied with.

Let $f(r)=0$ be an equation to be solved, and let a be an approximation to a root r of the equation.

Let us suppose that $f(r)$ is single-valued and analytic in a circle about a as a center and including r. Then

$$f(r)=f(a)+f'(a)(r-a)+\frac{f''(a)}{2!}(r-a)^2+\frac{f'''(a)}{3!}(r-a)^3+\cdots.$$

Hence, by the usual formulas for the reversion of series,[1] putting $z=f(r)-f(a)$, we get

$$\begin{aligned} r-a &= \left(\frac{dr}{dz}\right)_0 z+\left(\frac{d^2r}{dz^2}\right)_0\frac{z^2}{2!}+\left(\frac{d^3r}{dz^3}\right)_0\frac{z^3}{3!}+\cdots\left(\frac{d^nr}{dz^n}\right)_0\frac{z^n}{n!}+\cdots \\ &= A_1^{-1}z-A_2A_1^{-3}\frac{z^2}{2!}+(3A_2^2A_1^{-5}-A_3A_1^{-4})\frac{z^3}{3!}+\cdots A_1^{-1}\frac{d}{da}\left(\frac{dr^{n-1}}{dz^{n-1}}\right)_0\frac{z^n}{n!}+\cdots, \end{aligned} \tag{1}$$

where $A_k = d^kf(a)/da^k$ and $A=f(a)$. In practice it is necessary that the devel-

[1] See, for instance, Goursat, *A Course in Mathematical Analysis*, Vol. 1, §§ 189–190 (first edition).

opment in (1) should be rapidly convergent for the number of terms employed, and the conditions of convergence for the development make it possible to determine the degree of accuracy attained by using a given number of terms.

It will be observed that of all the derivatives A_k in (1), A_1 is the only one which has or can have a negative exponent, and hence the only one which must never have a zero modulus for any value of a as it approaches r. It will also be observed that in order that (1) may be rapidly convergent it is essential that modulus z should be as small as possible, and that modulus A_1 should be as large as possible.

Employing only the terms of (1) preceding the term containing z^4 and putting $z = f(r) - f(a) = -A$, since $f(r) = 0$ for a root r and $A = f(a)$, we get the practical working formula:

$$r = a - AA_1^{-1} - \tfrac{1}{2}A^2A_2A_1^{-3} - \tfrac{1}{2}A^3A_2^2A_1^{-5} + \tfrac{1}{6}A^3A_3A_1^{-4} \cdots, \tag{2}$$

which is quite well adapted to logarithmic computation for both real and imaginary values of the roots.

Should the degree of accuracy attained by using (2) not be sufficiently great when the first assumed value of a is employed, greater accuracy may be obtained by substituting the new value, and so on, until the required degree of accuracy is attained.

To more clearly indicate the use of the method the following examples will be useful.

Let $f(x) = x^3 + 2x - \sin x - 15 = 0$ be an equation, a real positive root of which, accurate to five decimal places, is to be found.

We know from Sturm's theorem, or can learn by trial, graphic methods or otherwise, that a real root lies between 2 and 3. Therefore assuming that $a = 2$, we get,

$$\begin{aligned}
A &= 2^3 + 2 \cdot 2 - \sin 2 - 15 &&= - \ \ 3.9093\\
A_1 &= 3 \cdot 2^2 + 2 - \cos 2 &&= + \ 14.4161\\
A_2 &= 3 \cdot 2 \cdot 2 + \sin 2 &&= + \ 12.9093\\
A_3 &= 3 \cdot 2 + \cos 2 &&= + \ \ 5.5839
\end{aligned}$$

Substituting in (2), we get $x = 2.244$. Again letting $a = 2.244$, finding new values of A, A_1, A_2, A_3, and substituting in (2), we get $x = 2.243666$, the accuracy of which is determined not by the convergence of (2) but by the values of the sine and cosine as given in the ordinary 6-place tables.

As a second example let us take the quartic equation

$$f(x) = x^4 - 3x^2 + 75x - 10{,}000 = 0,$$

which Merriman gives on page 34 of his *Solution of Equations*, 4th ed., to illustrate Lambert's method. Then taking, as he does, the approximate values of the roots to be the four roots of unity each multiplied by 10, we get

				Modulus	angle
$a=+\ 10$	$-\ 10$	$+10i$	$-10i$	Modulus	angle
$A=+\ 450$	$-1{,}050$	$+\ 300+\ 750i$	$+\ 300-\ 750i$	$\sqrt{652{,}500}$	$68°\ 11'\ 55''$
$A_1=+4{,}015$	$-3{,}865$	$+\ 75-4{,}060i$	$+\ 75+4{,}060i$	$\sqrt{16{,}489{,}225}$	$-(88°\ 56'\ 30'')$
$A_2=+1{,}194$	$+\ 1{,}194$	$-1{,}206$	$-1{,}206$	1,206	$180°\ 0'\ 0''$
$A_3=+\ 240$	$-\ 240$	$+\ 240i$	$-\ 240i$	240	$90°\ 0'\ 0''$
$A_4=+\ 24$	$+\ 24$	$+\ 24$	$+\ 24$	24	$0°\ 0'\ 0''$

Substituting in the terms of (2)

$a=$	$+10.00000$	-10.00000	$+10i$	$-10i$
$-AA_1^{-1}=$	$-\ 0.11208$	$-\ 0.27167$	$+0.18330-0.07728i$	$+0.18330+0.07728i$
$-\frac{1}{2}A^2A_2A_1^{-3}=$	$-\ 0.00186$	$+\ 0.01140$	$+0.00428+0.00402i$	$+0.00428-0.00402i$
$-\frac{1}{2}A^3A_2^2A_1^{-5}=$	$-\ 0.00006$	$-\ 0.00095$	$-0.00012-0.00033i$	$-0.00012+0.00033i$
$+\frac{1}{6}A^3A_3A_1^{-4}=$	$+\ 0.00001$	$+\ 0.00021$	$0.00000 \cdot 0.00000$	$0.00000\ \ 0.00000$
$x=\text{sum}=$	$+\ 9.88601$	-10.26101	$+0.18746+9.92641i$	$+0.18746-9.92641i$

Should still greater accuracy be required replace the above values of a by the values of x just found, and so on until the required degree of accuracy is attained.

A FORMULA FOR THE SUM OF A CERTAIN TYPE OF INFINITE POWER SERIES.[1]

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Introduction.

The problem to be considered is that of finding a definite, finite formula which will give the sum of any convergent infinite series whose terms are such that their numerators form an arithmetical progression of any order[2] and whose denominators form a geometric progression.

Since the nth term of an arithmetic progression of the kth order may be reduced to the form

$$a_n = b_0 n^k + b_1 n^{k-1} + \cdots + b_k,$$

our problem is to evaluate the expression

$$T = \sum_{n=1}^{\infty} \frac{b_0 n^k + b_1 n^{k-1} + \cdots + b_k}{ar^n}$$

in which the b's are independent of n. But this may be written

$$T = \frac{b_0}{a}\sum_{n=1}^{\infty}\frac{n^k}{r^n} + \frac{b_1}{a}\sum_{n=1}^{\infty}\frac{n^{k-1}}{r^n} + \cdots + \frac{b_k}{a}\sum_{n=1}^{\infty}\frac{1}{r^n}$$

[1] The author wishes to acknowledge criticisms and suggestions from Professors R. D. Carmichael and A. C. Lunn.

[2] See *Text Book of Algebra*, Chrystall, Vol. 1, page 484.